EFFECTIVE STRATEGIES AND ACTIVITIES
FOR
TEACHING NUMBER CONCEPTS AND OPERATIONS
TO
ELEMENTARY SCHOOL STUDENTS WITH LEARNING DISABILITIES:
A REVIEW AND ANALYSIS OF THE RESEARCH

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CHAPTER 1
INTRODUCTION

Purpose of the study

The purpose of this study is to investigate the numeracy development needs of elementary students with mathematical learning difficulties and determine the most effective teaching techniques and activities for providing instruction and intervention for students with mathematics learning difficulties. In order to do this, the author will explore some of the specific challenges facing children experiencing learning difficulties in mathematics, primarily in the areas of number concept development and number operations. Developing an understanding of the challenges facing students with mathematical learning difficulties will assist educators in determining the efficacy of interventions.

This study will attempt to answer four key research questions:

1. What are the key elements of number concepts identified as presenting the greatest challenge to students with mathematical learning difficulties?
2. What are the key components of successful mastery of number operations?
3. What general contributing factors to learning difficulty have the greatest impact for those with mathematical learning difficulties and how do these affect the acquisition of number concept and operations?
4. What instructional strategies and activities are most effective for assisting students with mathematical learning difficulties develop mathematical skills and concepts involved in the area of number concepts and operations in order to support effective numeracy development?
Problem underlying the study

It has become a widely recognized fact that 10 to 30% (van Kraayenoord and Elkins, 2004) of students in Britain, Australia and the United States have not mastered basic numeracy skills (Burns, 1992; Milton and Rohl, 2002; van Kraayenoord and Elkins, 2004; Woodward and Montague, 2002). In all these countries, there are now programs in place designed to improve mathematics instruction in the school, thereby ensuring improved numeracy of future generations. In the United States, the National Council of Teachers of Mathematics (NCTM) recognized that there were significant difficulties with mathematics education and responded with a document outlining the standards it believed were essential for effective mathematics education (NCTM, 1989). However, students with learning difficulties continue to be at risk for failure in mathematics (Montague, 1997).

For many years Canadian mathematics instruction for elementary aged students with mathematical learning difficulties has consisted primarily of math fact drills and practice of mathematical procedures such as addition and subtraction with regrouping, partial product multiplication, and long division. This approach to mathematics has left mathematically challenged students with little ability to apply mathematical skills to real life problems.

In response to changing needs in the job market, the scope of mathematics education has increased over time. Students are expected to cover a large range of mathematical subjects in a school year, and so topics are often covered quickly. Students with learning difficulties in mathematics, especially, suffer from the rapid pace in which topics are introduced (Westwood, 2003). For these students, such a situation has resulted
in a lack of conceptual and procedural knowledge as well as fluency, thereby increasing mathematical difficulties each school year possibly leading to math phobia. For these students, intervention often consists of pullout programs that emphasize mastery of basic facts and operations. This in turn can add to the student’s frustration level.

Despite the positive direction in which general mathematics education appears to be moving, educational changes have been slow to find their way into the special education domain. Students with mathematical learning difficulties continue to fall behind their peers and are less likely to acquire or master essential numeracy knowledge and skills. Research into numeracy development is relatively recent (within the last ten years), and special education has only begun to explore the importance of early numeracy skills for students with mathematical learning difficulties (Clarke, Gervasoni and Sullivan, 2000; Milton, 2000).

**Research methods**

This project will consist of a review of the literature, analysis and discussion of the current research on teaching practices in mathematics education, primarily in the area of mathematics learning disabilities or difficulties. Sources for the literature review will include peer-reviewed journals located in the online databases “Proquest”, “Ingentia”, “Expanded Academic ASAP”, “Ebsco”, and other electronic journals. The author will also consider other available texts and material created for the purpose of informing and guiding instruction.

**Significance of the study**

While researchers have begun to look at mathematics education for students with mathematical learning difficulties, this information has been slow to find application in
many regions. In the author’s province, an emphasis on numeracy has only recently become an area of focus for general education students. Many local teachers are unaware of possible effective interventions that could positively affect their students with mathematical learning difficulties. The purpose of this paper is to aid the author and interested readers in finding and implementing interventions that are helpful to elementary aged students.

Definitions

**Numeracy:** the ability to apply essential mathematical skills and knowledge as needed for everyday activities and problems (Gal, 2002; Gal and Stoudt, 1997; Hogan, 2002; van Kraayenoord and Elkins, 2004).

**Number concepts:** knowledge and awareness of number. This includes cardinality (numbers have size, order, and can be compared), place value (concepts of ones, tens, hundreds etc), awareness that quantities can be combined or separated, one to one correspondence, awareness that numbers can be compared and classified by characteristics (Liedtke, 1993; National Council of Teachers of Mathematics, 1989).

**Number operations:** at the elementary level, these are confined to addition, subtraction, multiplication and division of whole and rational numbers (Liedtke, 1993; National Council of Teachers of Mathematics, 1989).

**Learning difficulties:** This term will be used to refer to significant and long term difficulties in learning mathematical concepts and skills experienced by children of average intelligence. It will include the more common North American concept of mathematical learning disabilities, but allows for the inclusion of a broader range of students experiencing significant difficulties in mathematical learning (van Kraayenoord and

**Limitations and Delimitations**

Since this paper is limited in length, the scope of the topic has been limited to consideration of appropriate interventions for elementary aged students between grades two and six, experiencing learning difficulties in the areas of number concepts and operations. While student problem solving skills and assessment of mathematical knowledge are areas of concern for educators, this paper will not explore these issues as each alone contains enough breadth to encompass an entire study. While the author will attempt to outline a number of learning activities appropriate for developing selected number and operation concepts, it is beyond the scope of this paper to discuss or prove their effectiveness with individual student.
CHAPTER 2
LITERATURE REVIEW

Numeracy Definition

Numeracy development is becoming a common phrase in North America education in recent years. In 2001, the province of British Columbia identified early numeracy intervention as a key priority for educators. As part of this the government funded a research project which focused on three elements; identifying components of early numeracy (grades kindergarten and one), developing an assessment tool, and creation of an intervention program for teachers to use to aid young students in becoming mathematically literate (for further information see www.bced.gov.bc.ca/primary_program/). With such an emphasis on early numeracy, it is necessary to discuss what researchers and educators mean by the term “numeracy” (van Kraayenoord and Elkins, 2004). While there exists a wide number of definitions of numeracy, in general terms, it refers to the portions of mathematical knowledge and skill that are related to functioning within society (Hogan, 2002). In this respect, the elements of numeracy will vary depending upon the demands of a society. Therefore, changes in societal needs and goals will affect the definition of numeracy (Liedtke, 1991). With the increasing technological and social demands of society, it seems reasonable that mathematics education would be facing a significant challenge as it attempts to meet the changing demands of its society.

Numeracy is considered to be the “ability to understand and use the language of mathematics, to interpret everyday quantitative information, and to be aware of and use a range of strategies and number sense to solve problems” (Van Kraayenoord and Elkins,
Cass, Cates, Smith and Jackson (2003) state “students who wish to function successfully as adults in the 21st century must acquire an understanding of mathematical concepts, learn to reason and problem solve, and develop a positive attitude towards mathematics” (p. 112). Of particular importance, when considering the preceding view of numeracy, is the need for employees with adequate mathematical skills. The National Council of Teachers of Mathematics (NCTM) identifies seven essential skills needed by employers. They are:

- An ability to set up problems
- A knowledge of a variety of techniques
- An understanding of underlying mathematical features
- An ability to work with others
- An ability to see the applicability of mathematical ideas
- A preparedness for open problem situations

Since employment opportunities today require the above skills, it is clear that they contribute to an understanding of numeracy. In essence, numeracy consists of a conglomeration of flexibility, insightfulness, mathematical skills and knowledge, a positive attitude towards mathematics and a willingness to creatively apply these elements to life’s demands (Davis, 2001). While numeracy is clearly more than simply the ability to do math, without mathematical knowledge and skills there is no numeracy (Johnston, 1994; Hogan 2000).

Of particular concern is the group of students who demonstrate significant and long-term difficulties in learning, retaining and applying mathematical skills. These otherwise intelligent students, referred to as having mathematical learning difficulties, struggle with mathematical concepts and skills throughout their years of education and
often leave the education system without the essential numeracy skills needed to function adequately in society (Cass, Cates, Smith and Jackson, 2003; Gal, 2002; Whittaker, 2004).

**Current Research Trends**

The current focus in general mathematics education research appears to be on developing children’s ability to think about mathematics (Woodward and Montague, 2002). This is a reflection of the changing mathematical demands of life. The philosophy being promoted by many is labeled constructivism. The basis of this philosophy is that students construct their own meaning about mathematics based upon their observations and previous knowledge. Special education has been slow to embrace constructivism as a way to approach mathematics education (Montague and Woodward, 2002). A common view is that constructivism lacks structure and simply leaves students to discover information on their own with little or no direct instruction, and as an approach will most likely lead to even greater failure for students with mathematical learning difficulties (Woodward and Montague, 2002). There is little research available on the long-term effects of the constructivist philosophy on the educational success of students with mathematical learning difficulties but a great deal of criticism (Davis, 2001; Woodward and Montague, 2002). Instead, research has focused on structured approaches to teaching mathematical skills (Fuchs and Fuchs, 2001).

While special educators question the appropriateness of constructivist approaches, this must be balanced against past practice of placing emphasis upon rote learning of basic facts and computational algorithms, which has resulted in the generalized failure of students to develop conceptual understanding of fundamental mathematical ideas (Fuchs
and Fuchs, 2001; Liedtke, 1991; Woodward and Montague, 2002). Education has not produced numerate individuals when it has focused primarily on fluent retrieval of basic facts and the execution of computational procedures (Woodward and Montague, 2002). While such knowledge is necessary for numeracy, it has not proven to be sufficient (Goldman and Hesselbring, 1997). Addressing the imbalance between goals and interventions is necessary if students with mathematical learning difficulties are to become members of a numerate community.

**Mathematical Concepts Essential to Numeracy**

Research has identified three general goals for mathematics education, and these apply to all students, including those with mathematical learning difficulties. The primary elements identified are development of conceptual knowledge, procedural knowledge, and declarative knowledge (Goldman and Hesselbring, 1997; Liedtke, 1991). Declarative knowledge consists of a network of linked facts about mathematics (i.e. $3+4=7$), while procedural knowledge consists of skills and knowledge regarding rules, algorithms and steps needed to solve mathematical tasks (i.e. long division, multiplying fractions). Conceptual knowledge is a connected web of information linking relationships; it involves building understanding and links between pieces of information (Goldman and Hesselbring, 1997). Without all these elements present, students are unable to apply knowledge and skills to real problems, which equates with an absence of numeracy (Goldman and Hesselbring; 1997).

Westwood (2003) has restated this idea in general terms by identifying competencies that contribute to the development of procedural, declarative and conceptual knowledge. The five general areas identified are:
• The ability to comprehend relations, operations and concepts
• Development of procedural fluency
• Development of adaptive reasoning
• Development of strategic competency
• The presence of a productive disposition, the inclination to like math and to persevere to master it (Westwood, 2003).

This is what mathematics educators desire for their students. The issue to consider is what will help get students there. What specifics are worth spending our limited instructional time on?

**Numeracy Essentials**

Research into early numeracy identifies the following eight essential components:

- Understanding of principles and procedures related to counting
- Understanding of written symbols
- Understanding of the use of place value
- Understanding and ability to solve word problems
- Ability to translate between concrete, verbal and symbolic forms
- Ability to use strategies for fact recall when calculating
- Ability to estimate
- Establish memory recall for basic math facts (Dowker, 2001; Dowker, Hannington and Matthew, 2000).

The majority of these eight components can be divided into 3 categories: counting, place value concepts and number operations.

**Number Concepts: Counting**

Counting seems to be a very basic skill, yet it encapsulates many ideas. Often, older students with mathematical learning difficulties demonstrate difficulties with concepts dependent upon the ideas involved in counting (Frank, 1989; Liedtke, 1993). Counting is an important part of the development of children’s understanding of basic number concepts (Cardona, 2002; Frank, 1989; Numeracy Counts, 1998; Young-Loveridge, 2002). In order to count quantities, students must have developed one-to-one correspondence, classification, ordering (seriation) and a concept of comparison (Bos and
Correspondence involves assigning a unique label to each quantity, for example, “nine” means the same thing as “9”, which matches only with the quantity “*** *** ***”, no other label applies and the “9” label does not apply to any other quantity.

Classification involves students grouping objects and later, number ideas according to shared characteristics. Knowing that eight toys, eight fingers, the numeral “8” on a number line and the sum 3+5 all share a common characteristic of ‘eighliness’ may seem simple, but becomes a significant problem for a grade four or five child who does not understand that 358 shares a characteristic with 300+50+8, 3 hundreds 5 tens 8 ones, and 3 hundreds 4 tens and 18 ones.

Ordering or seriation involves the ranking of objects. In a simple form ordering can be seen in rote counting, later it can be understood as the placing of numbers in order by some shared characteristic, such as a number line, a pattern of skip counting or a list of multiples.

Comparison involves concepts such as greater than, less than and equal to. When linked with classification, these ideas provide a basis for an understanding of prime and composite numbers, as well as an understanding of fractions, decimals, multiples and divisibility.

These concepts, while developing initially in the pre-school and early primary years, must continue to develop throughout the elementary years, expanding to include new and broader concepts.

**Number Concepts: Place Value**

Place value concepts form an integral part of early number learning, and forms a
substantive part of number concept development (Wright, 2002). Students with mathematical learning difficulties need to master place value (Van Kraayenoord and Elkins, 2004). The “principal elements of our system of numeration are the base (ten), the symbols or numerals (0 to 9), the principal of place value and the decimal point. One of the most important outcomes of the elementary school mathematics program is the understanding of this system and especially the understanding of place value (Liedtke, 1991 p. 93). Bos and Vaughn (2002) identify seventeen numeration and place value concepts taught at the elementary school level. They are an understanding of:

- cardinality - understanding the face value of the digits 0 to 9
- grouping patterns - that quantities are grouped into sets of specified size
- place value - that the position of a digit determines its value
- place value (base 10) – each position has a value determined by a power of ten
- that there can only be one digit per place
- the linear and ordered structure of the system
- the placement and meaning of the decimal point
- how quantities can be regrouped
- implied zeros to the left of whole numbers and to the right of decimals
- that the value of a digit is the product of its face and place
- implied addition in the sense of expanded form
- the relative size of numbers
- the uniqueness of the names for digits
- how names for multi-digit numbers relates to its written form
- how place value periods relate to verbal naming
- how verbal naming of digits in the ones period is structured
- how digits in each period are read as if they were in the ones period, followed by the period name (Bos and Vaughn, 2002).

While many of these concepts seem abstract, the task of reading about, understanding, and expressing quantities requires some degree of understanding of all the above concepts. Attempting to apply a mathematical operation to multi-digit numbers and arriving at a meaningful conclusion requires a general understanding of place value. While appearing as a complex collection of knowledge, place value requires
conceptual understanding if it is to be applied and used in any meaningful way by students. Without this knowledge, it is clear to see how difficult even the simplest task involving numbers or calculation could be.

**Number Operations**

Students with learning difficulties in mathematics need to master the basic mathematical skills of addition, subtraction, multiplication and division (Van Kraayenoord and Elkins, 2004). In fact, much of the general math that is done daily, involves basic operations applied to everyday situations (Patton, Cronin, Bassett and Koppel, 1997).

Number operations involve a number of essential concepts. Developing part-whole thinking, an understanding that numbers are made up of other smaller numbers, is essential to mathematical growth (Young Loveridge, 2002). When students understand that six toys and two more toys can be grouped to make eight toys or that having four fish and two of them die leaves only two fish, they have a basic conceptual understanding of part-whole. Once children gain an understanding of this, they are able to use this information to find quantities. However, of primary importance is the development of the conceptual understanding involved in the four basis operations. Students should be able to talk about mathematical operations using everyday language, they should be able to dramatize or simulate actions, they should be able to demonstrate understanding using pictorial representations, both in creating their own and interpreting others, and they should be able to use numerals and symbols to describe events from their own experiences (Cardona, 2002; Liedtke, 1991). Conceptual understanding is the necessary element in having the ability to apply mathematical understanding to real life problems.
Researchers also highlight the need for efficient calculation and retrieval strategies. There is strong emphasis on the need for quick and efficient recall of basic math facts as this reduces the demands upon working memory and enables students to focus upon conceptual understanding and the relationships between varied ideas rather than becoming overwhelmed with calculation tasks (Bos and Vaugn, 2002; Geary, 2004; McLeod, 2001; Van Kraayenoord and Elkins, 2004; Woodward and Montague, 2002). For many children with mathematical learning difficulties, recall of facts is extremely challenging and so they rely on other strategies for finding answers (Geary, 2004). They generally use inefficient strategies, such as counting all objects, to find quantities (Geary, 2004; Liedtke, 1991; Kaufman, Handl and Thony, 2003). Hence, teaching strategies forms an important part of intervention plans (Liedtke, 1991; McLeod, 2001; Numeracy Counts, 1998; Van Kraayenoord and Elkins, 2004).

**Contributing Factors**

Researchers have identified several common characteristics of students with mathematical learning difficulties. In general, these students have difficulty in the areas of memory, abstract reasoning, and organization (Miller and Mercer, 1997; Steele, 2002; van Kraayenoord and Elkins, 2004; van Luit and Schopman, 2000). Memory problems are most noted in the areas of basic facts retrieval and working memory (Keeler and Swanson, 2001; Krosbergen, van Luit and Naglier, 2003; Woodward and Montague, 2002). The effect of this is many errors in calculations as students work slowly and have difficulty recalling steps in procedures. Students with mathematical learning difficulties
often have visual perceptual difficulties, such as loosing their place on a page or maintaining an organized page layout (Bos and Vaughn, 2002; Miller and Mercer, 1997). Many have attentional difficulties, which cause problems in sustaining attention to critical instruction and in completing essential steps in number operation algorithms (Bos and Vaughn, 2002; Miller and Mercer, 1997).

As well, students with mathematical learning difficulties often do not recognize when to apply knowledge or skills in real life situations (Bos and Vaughn, 2002; Goldman and Hesselbring, 1997). Students with mathematical learning difficulties use immature strategies for solving problems; compared to other children, their performance appears delayed but not cognitively different (Geary, 2004; Woodward and Montague, 2002). These students demonstrate significant math anxiety, learned helplessness, low self-concept, and a lack of persistence (Bos and Vaughn, 2002; McCoy, 1995; Miller and Mercer, 1997). This is not surprising, as these students often struggle with mathematics for years, becoming more and more discouraged and frustrated as the mathematical content moves further and further from their grasp.

A significant contributing factor not related to student ability is poor teaching. There are a number of criticisms directed towards poor mathematics instruction. These criticisms include failure to ensure relevant prior knowledge, introducing concepts too quickly, providing too little opportunity for guided practice and failure to present steps and strategies clearly (Engelmann, Carnine, and Steely, 1991; Westwood, 2003). General education students struggle with mathematics under such circumstances, but students with mathematical learning difficulties have little chance to succeed when faced with such circumstances.
The impact of these difficulties outlined above have their effect over a broad area, including computation, mathematical reasoning, mathematical language, generalizing of strategies, and problem solving (Cass, Cates, Smith and Jackson, 2003; Miller and Mercer, 1997; van Kraayenoord and Elkins, 2004). These students struggle with mathematics, demonstrating little growth in understanding unless appropriate and timely interventions are put into place to help them learn despite their difficulties.

**Research Recommendations for Intervention**

Effective instruction is an important issue in addressing mathematics learning difficulties (Milton and Rohl, 2002; van Luit and Schopman, 2000). Unfortunately, there is a strong link between difficulties in mathematical learning and poor instruction. Westwood (2003) identifies a number of instructional factors including insufficient or inappropriate instruction, too rapid a pace for instruction, too little structure in instruction, the introduction of abstract symbols too soon, the removal of concrete and visual aids too early, and too little review and revision. While educators and researchers can and have identified a number of problem areas in instructional practice, “there is considerable debate as to what might constitute the instruction or intervention for students with [learning difficulties] in numeracy” (van Kraayenoord and Elkins, 2002, p. 35).

Research into effective intervention strategies for students with mathematical learning difficulties can be divided into two general categories. The first focuses on general teacher strategies which may have application to a variety of subject matters and student achievement levels. These address the quality of general teacher practice and are beneficial for addressing the specific needs of students with mathematical learning difficulties, but are not unique to this population. The second category focuses primarily
on specific interventions for a small set of specific skills (such as strategies for retrieval of basic facts). Teaching the traditional algorithms is one of the most common instructional activities for students with mathematical learning difficulties; mastery of procedural steps is emphasized, and student growth is measured in minute steps from isolated assessment tools whose tasks bear no relation to real life situations (Tournaki, 2003). Unfortunately, students can learn these skills without having developed any conceptual understanding, without having developed knowledge that would lead to numeracy (Woodward and Montague, 2002).

Poor conceptual knowledge greatly affects a student’s ability to use mathematical skill and knowledge in practical and useful ways. The focus of much of the past research into instruction for students with mathematical learning difficulties has been on developing declarative and procedural knowledge rather than on building conceptual understanding (Woodward and Montague, 2002). Most students with mathematical learning difficulties are inundated with procedural knowledge, and spend much of their mathematics instruction time on the ‘how to’ portion of mathematics (Liedtke, 1991; Woodward and Montague, 2002). Their time is spent learning the steps in multiplication, division, addition and subtraction, but is not balanced with ‘when’, ‘why’ and ‘how’ to apply the mathematical skills they are learning. Current research in mathematics education recommends a focus on teaching concepts rather than continuing with an overemphasis on procedures (Smith and Gellar, 2004). Teachers who emphasize the building of connections between procedures, real life situations and mathematical ideas have classes which progress more than those whose focus is primarily on procedure instruction (Askew, Brown, Rhodes, Johnson and Wiliam, 2004). However, this must be
balanced against the research that emphasizes the failure of constructivist classrooms to meet the needs of all students (Fuchs and Fuchs, 2001; Westwood, 1996). Numeracy does not result without a balance between procedural, declarative and conceptual understandings. If any one of these three elements is missing, students do not have mathematics that they can use.

**Recommended Practices**

Mastriopieri, Scruggs and Shiah (1991) identified a number of practices which were noted by research as being effective in teaching mathematics to students with learning difficulties (as cited in Miller and Mercer, 1997). The list includes using demonstration, modeling, providing reinforcement for fluency building, using a concrete-to-abstract teaching sequence; combining demonstration with a permanent model, using verbalization while solving problems, teaching strategies for computation and problem solving, and using peers, computers and other technology as alternative delivery systems (Miller and Mercer, 1997). Other educators concur, but may have added or deleted several elements (see Cass et al, 2003; Fuchs and Fuchs, 2001; Miller, Butler and Lee, 1998; Smith and Gellar, 2004; Steele, 2002; Westwood, 1996).

The current research, taken as a whole, emphasizes a balanced approach that does not favor one form of knowledge over any other. Students need explicit teaching of procedures, hands on manipulatives and demonstrations for developing declarative knowledge tied together with conceptual development involving problem solving in familiar contexts, building links with familiar concepts and developing an understanding of the bigger ideas (Booker, 1999; Carnine, 1997; van Kraayenoord and Elkins, 2004; Westwood, 2003; Woodward and Montague, 2002).
Specific Strategies to Aid in Learning Number Concepts and Operations

Of primary importance to numeracy development is conceptual understanding. There are technological tools (calculators, computers) that are capable of accurately carrying out all mathematical procedures. However, in order to use these tools effectively, students must recognize when and where to use procedures and apply knowledge. The following elements of instruction are associated with conceptual development.

**Familiar Contexts**

A useful tool for developing an understanding of the actions associated with the four basic operations of addition, subtraction, multiplication and division is the use of story (word) problems. Story problems create a context and meaning for procedures. Students who have the ability to translate a story problem into a representative diagram or algorithm indicate conceptual understanding (Liedtke, 1991). Having students create story problems, draw pictures or use manipulatives to show what happens in teacher created story problems, and then work together to make and solve one another’s stories problems will reinforce and build a conceptual understanding of the four basic mathematical operations (Liedtke, 1993).

**Manipulative Materials**

Researchers recommend the use of manipulative materials when working with students experiencing mathematical learning difficulties. The use of manipulative materials has been shown to increase mathematical performance and to help facilitate conceptual understanding (Cass et al, 2003; Fuchs and Fuchs, 2001; Miller, Butler and
Lee, 1998; Westwood, 2003). Manipulative materials include base ten blocks, pattern blocks, fraction bars and circles, connecting blocks, realistic play money, counters and many other items that can be used to illustrate mathematical ideas or make links between concepts in a concrete way.

**Developing Basic Facts Retrieval Strategies**

The use of manipulatives aid students in visualizing an operation or idea. However, once students progress to multi-digit calculations, they benefit by having knowledge of and rapid recall of basic information (i.e. $9+6=15$, $4\times7=28$). Students who do not develop this find themselves at a disadvantage because they must devote more of their attention to figuring out answers rather than learning procedures or underlying concepts (Woodward and Montague, 2002). Most students demonstrating fact retrieval difficulties rely heavily on finger or verbal counting to find answers (Geary, 2004; Liedtke, 1991). In fact, most use a counting all strategy (i.e. $3+8$ is counted 1,2,3 and then 4,5,6,7,8,9,10,11) (Woodward and Montague, 2002). In order to address this problem, students require strategy instruction (Tournaki, 2003). Their peers have progressed forward to more elaborate strategies that contain a wealth of conceptual sophistication, such as ‘min’ counting (understanding that $2+9$ is equal to $9+2$), counting on (thinking $9+2$ as 9 and then 10, 11 rather than 1, 2, 3, 4, 5, 6, 7, 8, 9, and then 10,11), using doubles ($7+8=7+7+1$) or using ‘adding to 10’ ($7+8=7+(3+5)=10+5$). The student with limited counting strategies fails to notice these helpful relationships and patterns within number concepts (van Kraayenoord and Elkins, 2004; Woodward and Montague, 2002).

**Games**
Once students have learned strategies and procedures, game format activities can be utilized to provide practice and meaningful use for their skills. As well, mathematical games can encourage mathematical talk. Talking about mathematics has an important role in helping students make sense of mathematics, helping them organize and clarify their thoughts, building confidence and encouraging recall (Booker, 1999; Liedtke, 1991; Numeracy Counts, 1998). As well, games are motivating and are powerful tools for changing student’s attitudes towards mathematics (Liedtke, 1991). It is also possible to engage students of various levels of proficiency to work together actively. This active involvement combined with peer modeling and an appropriate level of learning activities leads to a greater incidence of learning (Reys, Suydan, Linquist and Smith, 1998).
CHAPTER 3
ANALYSIS AND DISCUSSION

There is an obvious need to change some of what teachers do when involved in the teaching of students with mathematical learning difficulties. When consideration is given to the requirements of everyday existence and the increasing mathematical demands by many employers, it becomes clear that many students with mathematical learning difficulties will not graduate from secondary school with mathematical competency in everyday mathematical tasks. There is strong support in current literature for implementing changes in instruction for students with mathematical learning difficulties (Burns, 1992; Milton and Rohl, 2002; Westwood, 2003). There is recognition that mathematics instruction for these students has not always been effective. As a result, without thoughtful, informed changes to instruction, students experiencing mathematical learning difficulties may not develop the skills or knowledge necessary to become numerate individuals.

Development of Conceptual Understanding

One idea highlighted in the research discussed previously, is that of developing conceptual understanding. Students lacking in conceptual understanding, may well have mastered many mathematical procedures and have a variety of mathematical skills, yet have such a limited understanding of the underlying ideas that they cannot recognize when to use their knowledge (Kaufman, Handl and Thony, 2003). Knowledge that cannot be used has little or no value. Many students with mathematical learning difficulties have spent most of their mathematics instruction time learning to correctly add, subtract,
multiply, divide, yet cannot recognize a problem in any form other than the one that they have practiced. Memorizing and being able to correctly recall the correct algorithm for long division or multi-digit multiplication is useless when one cannot solve a simple problem such as “I make $137.50 per week working after school, and I want to go on a trip to Fiji with my friend and his parents next Christmas. I will have to pay for the airfare myself. The airfare to Fiji will cost $3987 plus taxes. How many weeks do I need to work to save the money? Do I have enough time to save the money? How much money do I have to save each week between now and then in order to pay for the ticket?”

Developing a conceptual understanding of mathematical ideas means being able to consider real mathematical problems and recognize within them some element that matches with a mathematical experience, model or idea, which is linked in various ways with other mathematical ideas or knowledge. Tools such as calculators and computers are able to perform the majority of calculations quickly and easily, but are useless to students if they do not recognize mathematical ideas within problems. Without conceptual understanding, students will not know which buttons to push to find an answer, and in the end will not recognize whether or not their solution is reasonable or correct.

**Contributing Factors**

Many students with mathematical learning difficulties share common characteristics that contribute to their learning difficulties (Keeler and Swanson, 2001; Kroesbergen, van Luit and Naglieri, 2003; Miller and Mercer, 1997; Steele, 2002; van Kraayenoord and Elkins, 2004; van Luit and Schopman, 2000; Woodward and Montague, 2002). Difficulties in memory retrieval, working memory, abstract reasoning and organization
contribute to difficulties in learning mathematics, particularly when taught as a series of skills to be memorized. Memory retrieval problems interfere with memorization of math facts, especially when taught as rote facts. Working memory difficulties interfere with remembering the steps in calculations. It also results in many errors in calculations because students work slowly, spend large amounts of time trying to figure out answers to embedded calculations (i.e. 274 x 46 requires knowledge of 4x6, 7x6, 2x6, 4x4, 7x4, and 2x4), and then lose their place in the more involved calculation because all of their working memory resources were devoted to solving simpler questions. Students are not able to attend to, think about, and link ideas and knowledge together to develop an understanding of broader concepts if their attention is focused on trying to solve simple arithmetic calculations.

Rapid instructional pace, as noted by Westwood (2003), may contribute to memory difficulties as students are presented with many new ideas and skills quite quickly, and so have little time to become proficient or to assimilate or accommodate new information into their current understanding. The result is that students are forced to work with incomplete, unrelated, individual pieces of information, rather than accessing chunks of knowledge tied together by conceptual understanding.

Abstract reasoning difficulties also contribute to poor conceptualization of mathematics. Abstract reasoning is not a strength of children throughout most of their elementary years, yet researchers suggest that the difficulty is greater for students with mathematical learning difficulties (Miller and Mercer, 1997; Steele, 2002; van Kraayenoord and Elkins, 2004; van Luit and Schopman, 2000). When mathematical skills
and concepts are presented as abstract ideas, these students will have great difficulty learning the concepts. The use of manipulative materials and ensuring that problems are set in everyday contexts will help educators to avoid a tendency to reduce mathematics to its most abstract form.

Many students also have difficulty with simple, non mathematical tasks such as copying down text book questions correctly as they have difficulty chunking information and so are constantly shifting focus from one place (the textbook or blackboard) to another (the paper or book they are copying to). They often lose their place on the page and either spend time finding where they were, or do not notice they are looking at a different space and copy down questions incorrectly. Significant difficulties are also noticed in the area of neatness and organized page layout, especially in relation to multi-step, multi-digit calculations. Students do not lay out calculation questions in a vertical format so that place value is apparent, and so end up adding digits from different places together or missing digits altogether. All of this contributes to increasing difficulties in the areas of math anxiety, learned helplessness and a lack of persistence. Students quickly learn that no matter what they do the answer is always wrong, and in turn, protect themselves from failure by avoiding situations that could result in failure (McCoy, 1995). It is much easier to refuse to learn than to fail to learn. This may be a contributing factor in the attentional difficulties noted by Bos and Vaughn (2002) and Miller and Mercer (1997). While some students may have attention difficulties, it is also possible, that for many students with learning difficulties, attending to teacher lessons and instruction is not seen as a valuable activity, and so such students may not chose to engage in learning during instruction.
As well, students with mathematical learning difficulties often do not recognize when to apply mathematical knowledge, but as discussed earlier this may be more linked to poor conceptualization of mathematical ideas than an inherent difficulty contributing to mathematical difficulties. If students have not understood the mathematical ideas they have been taught, it seems reasonable to assume that they would not be able to use them successfully to solve problems. Poor learning could also contribute to the immature strategies demonstrated by the majority of students with mathematical learning difficulties. A student will miss relationships and patterns when engaged in activities that he or she does not understand. Perhaps, many of these issues are more symptoms of poor instruction than they are causes of poor learning.

Instructional Principles

While there is a large body of literature available to inform teachers of possible teaching methods and strategies, the goal of developing numeracy must always be kept in mind. The primary goal of instruction must be to develop conceptual understanding. Procedural fluency and basic fact memorization may be important skills, but without conceptual understanding, which enables the application of mathematical ideas to real problems, these skills are virtually useless. Individuals who understand mathematical ideas but have poor computation skills can use a calculator to circumvent their weakness, but individuals who do not understand the ideas have no tools at their disposal.

Educators can develop conceptual understanding by providing demonstration and modeling of procedures and ideas within familiar contexts. Most students understand shopping, money, going on trips, and game scores, amongst a host of other ideas. All of
these can be utilized to create problems to which the student can relate. Consider the computational exercises “27.95 + 19.37, “52.47-47.32”, and “5.15+1.99”. Students engaged in completing these questions only demonstrate that they can perform simple calculations. There is much more meaning and potential for concept development in the story “John and Jim decided to pool their money to buy a mother’s day a gift. Their dad has promised to give them any remaining money they need. John has $27.95 and Jim has $19.37. The gift they want to buy costs $52.47, including taxes. Do they have enough money to buy the gift? Do they have enough left over to buy a card for $1.99? If not, how much money do they need to get from their dad?” Both the computational exercises and the story problem involve the same set of calculations, however the story problem contains ideas about combining amounts (addition) and comparing amounts (subtraction), which help students to develop an understanding of what addition and subtraction might mean, and how one might recognize them in real life situations. The calculation questions only demonstrate whether the student can correctly line up digits, place the decimal point appropriately, and apply regrouping skills; the student need have no understanding of what is being done. Students can utilize play money to solve the story problem, they can work with partners, they can draw pictures, in fact, are free to be as creative and inventive as needed in order to find the answers to Jim and John’s questions.

There are many mathematical concepts that are necessary to an understanding of number and number operations. Students need to develop mental images of the four basic operations, creating an understanding of what action each operation entails (Cardona, 2002). As well, relationships between numbers, such as quantity comparison, number
attribute awareness, pattern and place value are essential elements of working with numbers, learning basic facts and developing procedural fluency. Students who have developed an understanding of the relationships between numbers are able to use that understanding in many ways, including developing strategies for the quick recall of basic facts.

Developing conceptual understanding is essential to numeracy, but procedural knowledge should not be abandoned. Conceptual knowledge is strengthened as it is linked to a broader array of knowledge, and including procedural knowledge in that web can only enhance student numeracy. Despite the availability of calculators, knowledge of mathematical procedures is valuable to have, and students should have proficiency in finding the answers to calculations involving the four basic operations of addition, subtraction, multiplication and division. There has been a substantial amount of research conducted in the area of instruction and three elements that seem to be agreed on by most researchers are the use of a concrete-to abstract teaching sequence using manipulatives and pictures, explicit teaching of procedures through demonstration and modeling, and provision of ample guided and independent practice (Booker, 1999; Cass, Cates, Smith and Jackson, 2003; Fuchs and Fuchs, 2001; Miller, Bultler and Lee, 1998; Miller and Mercer, 1997; Smith and Gellar, 2004; Steele, 2002; Westwood, 2003). This involves teaching specifics related to a operation procedure, demonstrating it through the use of manipulative materials such as base-ten blocks or counters and drawing representative pictures to model the process. The next step involves providing the student with time to complete a number of questions with teacher guidance and reinforcement, and lastly, when it is likely that the
student can be successful independently, providing opportunity for practice. Students work with concrete manipulative materials and learn the abstract procedures through their understanding of the concrete models they use. Students are encouraged to use manipulatives, pictures or diagrams and graphic organizers such as lists of steps and checklists until their understanding of, and competency with, the abstract procedure is fluent (Miller, Bulter and Lee, 1998; Steele, 2002).

Practice of mathematical skills and procedures is an important part of developing fluency and exploring ideas. Most students with mathematical learning difficulties have experienced very little enjoyment or satisfaction in learning mathematics. They have experienced too much failure, and are unwilling to risk further failure and discouragement by engaging in practice. As a result, motivating these students can be challenging. Games are one effective way to provide practice with many concepts and skills, because they are fun, and there is very little risk in being wrong. Games also provide an opportunity for students to share their understanding of math with one another as they challenge responses and share knowledge and skills between themselves.

Effective teaching of mathematics ensures that students have conceptual understanding of operations and number concepts through story problems and concrete demonstrations. This understanding is then used to build knowledge of operational procedures. Development of strategies for recall of basic facts relies on an understanding of the relationships between numbers. Helping students to master the basic concepts of number, place value and operations may not be sufficient to ensure numeracy, but this does form an essential part of early mathematics learning, providing the foundation for most
Essential Concepts in Number and Number Operations

While there are an infinite number of mathematical concepts that lead to increased numeracy, it is neither possible nor desirable to identify all of them nor discuss the ways in which they can be effectively taught. Of far more value is the identification of the most essential elements of numeracy. It is of much greater benefit to the students to identify the information most worth knowing, and ensure prioritization for the effective use of instructional time (Woodward and Montague, 2002).

Number Concepts

The most important concepts necessary for developing number concepts are a well-developed understanding of correspondence, classification, ordering (seriation), and comparison (Bos and Vaughn, 2002; Van Luit and Schopman, 2000). Understanding correspondence means that the student understands that each number is unique and represents a specific quantity. In early years one-to-one correspondence is emphasized, but most older students have consolidated one to one correspondence of smaller numbers. Their difficulties lie with larger numbers, and the various ways of representing them. The number two hundred thirty-eight has a unique name “two hundred thirty-eight”, symbol “238”, and can be represented by a number of configurations of base-ten blocks. Correspondence also forms a basis for the understanding that quantities can be combined to form other, unique quantities. This idea of correspondence is essential for counting, understanding place value, making change, working with measurement, and understanding number operations.
Classification is grouping objects and later, number ideas according to shared characteristics. Understanding 3 fish, 3 cars and 3 fingers all represent the same idea is essential for developing correspondence. Understanding that 14, 24 and 74 are similar because they share a common element, four ones, is important for developing place value concepts. Understanding that 4, 8, 12, 16, and 24 are similar because they are all multiples of four helps to develop basic facts recall and aids in procedural fluency with number operations. Understanding that 4x5, 40x5, and 400x5 share common characteristics because they can all be represented by 5 groups of 4 base-ten blocks helps to develop an understanding of patterns in number, but also helps develop an understanding of the multi-digit multiplication algorithm.

Ordering (seriation) requires students to rank objects in some order. While this could be as simple as reciting the number names to 10, ordering also involves an understanding that numbers can be ordered in many ways depending on shared characteristics. Understanding patterns in place value and in how numbers are written requires ordering. Understanding skip counting patterns, multiplication patterns, or geometric series, even recording quantities in order by size all require ordering.

Comparison involves concepts such as greater than, less than and equal to which are necessary to the understanding of subtraction and addition. Students who subtract one number from another, yet arrive at and are satisfied with an answer greater than they started with have not yet developed comparison of number. Understanding that numbers have size relative to each other is important when comparing numbers, but is an important part of estimation as well. Comparison is not limited to which number is greater or less, but
also looks at ideas such as rounding, which forms a basis for estimation skills. Estimation skills are an important part of everyday life, whether it is an estimate on the cost of car repairs, household remodeling, or how much the groceries in the shopping cart are likely to cost. Estimation also forms the basis for deciding the “correctness”, sensibleness, or value of a mathematical answer. Estimation allows students to utilize a conceptual understanding of number ideas to monitor successful use of calculation strategies (NCTM, 2000). Well-developed concepts about the relative size, the sameness and differences between numbers enable a student to develop the ability to estimate meaningfully.

**Place Value**

Place value concepts form the basis for the number system that is used in North America. As students progress through the grades, they are expected to expand their understanding to include numbers as large as ten million and as small as one thousandth. However, understanding of place value concepts is the same, regardless of the size of the number. Students who understand fully the concepts of place value for numbers between 0 and 1000 have a strong foundation upon which to build an expanded understanding of decimals and very large number values. Once a student understands how the base-ten number system works for numbers less than one thousand, their conceptual understanding can be expanded to understand both smaller and larger numbers.

The most important concepts necessary for place value are understanding that the digits in numbers each represent a different quantity (i.e. 1234 represent 1 thousand, 2 hundreds, 3 tens and 4 ones), understanding the ten to one ratio between the digits, understanding that the position of a digit determines its value, understanding that there can
only be one digit per place, understanding implied addition in the sense of expanded form (i.e. 1234=1000+200+30+4) and understanding relative size of numbers (Bos and Vaughn, 2002). These four concepts are easily demonstrated using concrete manipulative materials, and using numbers already familiar to the student. The majority of other concepts, while important to the understanding of place value, are abstract in nature, and are successfully learned only when the basis provided by these four concepts is in place.

Understanding that digits in different places represent different values is essential to developing a clear understanding of addition, subtraction and long division. Combining this knowledge with an understanding that the position of a digit determines its value is essential for the reading and writing of numbers and number symbols. Place value is also important for fluency in operational procedures, and becomes essential when regrouping of quantities is introduced to students. Without a good grasp of the ten to one relationship between places, students do not develop an understanding of regrouping in any of the four operations. An understanding of expanded form is essential to understanding the partial products multiplication algorithm commonly taught in British Columbia’s classrooms. Understanding these concepts, in turn, leads to an understanding of the relative size of numbers, which ties directly to concepts involved in the comparison of numbers.

Number Operations

Students with learning difficulties in mathematics should develop proficiency in the basic mathematical skills of addition, subtraction, multiplication and division. This involves teaching procedures for each of the basic operations, from the lowest levels of adding and subtracting numbers to 9 and basic multiplication and division facts to 9x9,
through regrouping, partial products, long division, and working with decimal fractions.

In order to develop a clear understanding of these concepts students need to develop part-whole thinking, which is an understanding that numbers are made up of other numbers (Young-Loveridge, 2002). Students should also be able to talk about mathematical operations using everyday language. They should be able to dramatize or simulate actions and they should be able to demonstrate understanding using concrete and pictorial representations, both to tell their own math stories and to interpret math stories created by other individuals. Lastly, students should be able to use numerals and symbols to describe their own mathematical experiences (Lietdke, 1991). All of these activities work towards developing a conceptual understanding of each mathematical operation, and in combination with procedural fluency, result in an increased level of numeracy.

In order to develop efficient calculation, students need quick and efficient retrieval of basic math facts (Geary, 2004; McLeod, 2001; Van Kraayenoord and Elkins, 2004; Woodward and Montague, 2002). This helps reduce the demand on working memory and results in greater attention placed on the skill to be mastered and causes less frustration for the students. Awareness of number concepts, patterns and development of strategies such as counting on, grouping for ten, and adding doubles, reduces the need for rote memorization, but increase the likelihood of quickly performed and successful basic calculations. Therefore, providing instruction in these strategies through demonstration, guided practice and independent practice should form an important part of intervention plans for students having difficulty in basic facts retrieval (Lietdke, 1991; McLeod, 2001; Numeracy Counts, 1998; Van Kraayenoord and Elkins, 2004).
Specific Interventions to Develop Number Concepts and Operations

As discussed earlier in this chapter, there are a number of effective instructional strategies that help to develop conceptual and procedural knowledge. These include linking to familiar or prior knowledge, breaking concepts down into small steps, modeling and demonstrating by the teacher, providing opportunity for guided practice, teaching strategies, availability and use of manipulatives, and working within familiar contexts (Carnine, 1997; Cass et al., 2003; Miller et al, 1998; Smith and Gellar, 2004; Steele, 2002; van Kraayenoord and Elkins, 2004). The primary purpose of this section is to outline specific activities and intervention approaches that fulfill the criteria of the above instructional practices.

Interventions for Developing Number Concepts involved in Counting

Appropriate materials for teaching number concepts are those that clearly illustrate relationships between quantities. Base ten blocks, number lines, counters and hundred charts all are helpful for students.

All activities should be designed to ensure that appropriate instruction is provided. The following elements will be utilized to assist students in developing concepts related to counting.

Linking to familiar or prior knowledge: Students should be able to rote count to some quantity between twenty and one hundred prior to developing other counting skills as this will form a reference point for further learning. For most students in grades two to six, familiarity with numbers generally extends to numbers greater than 20. When students begin to work with numbers greater than one hundred, place value concepts should be
developed.

*Breaking concepts down into small steps:* Counting skills should be taught using smaller groupings of numbers so that students are not overwhelmed by having to understand and master too much information at one time. Learning of new skills or ideas should begin with numbers that the student is very knowledgeable about, as this enables the student to focus on the specific knowledge to be learned rather than developing proficiency with unfamiliar numbers. Depending on the students knowledge of numbers it may be necessary to begin working with numbers less than twenty, then moving on to numbers between twenty and fifty, then numbers to one hundred and so on.

*Modeling and demonstrating by the teacher:* All concepts and activities should be modeled for the student using manipulative materials and pictures before the student begins to practice. Students may need to be taught how to use manipulative materials such as number lines and base-ten materials.

*Providing opportunity for guided practice:* The teaching of a new concept should be accompanied by ample practice time for students. Ensuring mastery before progressing to larger numbers or increasingly complicated relationships is essential if students are to experience success. Understanding of concepts does not always come quickly, and moving beyond a student’s level of understanding or proficiency is sure to cause frustration.

*Working within familiar contexts:* Number concepts are often presented in an abstract way even when manipulatives are used. Asking a student to state the greater number in a pair, or order a set of random numbers is quite abstract. However, placing the same numbers
into a familiar setting through story provides a context for students and may make the task more meaningful. Providing a familiar context for exploring number ideas is helpful for students as they try to make meaningful connections between what they already know and what they are learning.

**Activities to develop understanding**

*Correspondence:* To develop an understanding of correspondence begin by using numbers with which the student is familiar. Model building a quantity with counting objects. Ask the student if he or she knows another way of showing the same number. Some students may count out objects and arrange them in a different formation. If they do so, arrange the teacher’s quantity in the same formation and then ask the student if the quantities are the same. Move the manipulatives about and reorganize them as needed to show that quantities are the same. Model naming and writing of number symbols for the student. Then provide the student with practice in building, recording and naming quantities until he or she is skillful and accurate within the restricted group of numbers. Once they have understood the concept with familiar numbers, expand their experience and knowledge by working with small groups of numbers of increasing quantity.

*Classification:* Classification can be applied to many aspects of mathematics from geometry to patterns in multiplication charts. The scope of this paper does not allow for summarization of all activities to cover all possible eventualities and so will be restricted to developing beginning classification skills.

To develop an understanding of classification, begin with paper plates or trays and a collection of objects of various shapes and colours (small toys are often a good choice).
Create a grouping of items according to some characteristic (shape, size, colour, etc.). Tell the student that you have chosen to put these objects together because they all have something in common. Have the student tell you what he or she thinks the common element is. Assist the child to be successful and model thinking aloud reasoning skills such as “they all aren’t the same colour so that isn’t it. There is a truck with big wheels, a bike with red tyres and a big yellow tractor. They all have wheels, maybe having wheels is what is the same.” Have the child build groups for the teacher to guess the sorting rule. Once the student is successful with this simpler task, he or she can be introduced to numerical classes. Have 2 or 3 paper plates, each with the same number of items (less than 6), but with no other shared characteristic. Tell the child that there is something the same about each group of objects. The student may require guidance and prompting to notice that the shared characteristic is quantity. Continue creating groupings of objects in different ways for the child to determine classification and provide opportunities for the student to challenge the teacher as well. Once students understand that objects and quantities can be grouped in different ways, and are able to demonstrate reasoning and logic skills with the simpler activities, they can be introduced to other increasingly challenging ways of classifying numbers. (Adapted from Liedtke, 1993).

**Ordering:** Developing of an understanding of ordering (seriation) can be fostered by beginning with a small selection of objects (toys, classroom items, etc.), and choosing two that differ by some characteristic (height, length, width, weight, etc.). Ask the student how they differ, and then discuss how they could be put in order (shorter then taller, lighter then heavier). Add other objects that can be ordered by that single characteristic and discus
with the student where each is to be placed. When the student can independently complete such tasks, he or she can be given groups of objects to place in order by some characteristic. The student can then be given small groupings of quantities to compare. Build each number and discuss which is more, less, etc. and have the student place them in order by quantity. A number line can be created, and the child can work with small groups of numbers at a time to fill in missing values on the number line. As the student shows understanding, the quantities to be ordered can involve larger numbers. (Adapted from Liedtke, 1993)

Comparison: Comparison follows naturally from ordering. Once the student has developed an understanding of the ordering of numbers, it becomes a much simpler task to teach comparison. Student will need to be taught the vocabulary “greater than” and “less than” through teacher modeling. Begin by presenting the student with two familiar numbers. Ask the student to build each quantity with manipulative blocks. Ask them which quantity is “smaller” or “less than” the other and which is “more” or “greater”. Model the placement of the numbers on a number line. Have the student work with a variety of increasingly larger numbers as they demonstrate success and understanding of the idea of comparison.

Interventions for developing Place Value Concepts

Place value can be successfully taught when a student has an understanding of both one-to-one and many-to-one correspondence. The student should have developed enough fluency with numbers to understand that the quantity 30 can be represented in more than one way: as 30 counters, as 30 “unit” blocks and as 3 “tens” blocks.
Appropriate materials for teaching place value concepts are those that clearly illustrate the ten to one relationship between the digits (Liedtke, 1993). Base ten blocks are one manipulative that can be used effectively to teach place value as they clearly show the relationship between ones, tens, hundreds and thousands. However, they are limited because they do not illustrate place value for numbers exceeding nine thousand nine hundred ninety-nine. Play money can also be used, but successful use is dependent upon students having a well-developed understanding of money concepts.

All activities should be designed to ensure that appropriate instruction is provided to students. The following elements will be utilized to assist students in developing concepts related to place value.

*Linking to familiar or prior knowledge:* If students have already developed some understanding of basic money, such as counting pennies, dimes and dollars, this knowledge can be used to help them understand the more abstract concepts involved in teaching place value. Canadian money utilizes the base-ten number system, and so many students will be familiar with one-dollar coins and ten-dollar bills. In order for students to understand place value, they must have a well-developed sense of numbers to at least one hundred. A student’s knowledge about numbers, how they are written, how they are named, and the ability to concretely represent quantities can be utilized to introduce new concepts in place value.

*Breaking concepts down into small steps:* Students should be introduced to place value concepts systematically. Their initial introduction should be limited to numbers less than one hundred. This enables the student to develop a clear understanding of the relationship
between ones and tens. Once the student is proficient and has a strong conceptual understanding of place value ideas, his or her knowledge can be expanded to include hundreds and thousands, then millions and quantities less than one.

*Modeling and demonstrating by the teacher:* The teacher should model building, recording and working with all aspects of place value before asking students to do so. This should include modeling the use of manipulatives, pictures, diagrams, vocabulary, and various ways of showing increasing skills and concepts. As well, providing a visual structure to support students and modeling its use can be valuable to students. A place value chart posted in the classroom serves as a reminder of vocabulary and reminds students of what the teacher has previously modeled. As well, students benefit from verbal modeling, hearing the teacher model reasoning aloud.

*Providing opportunity for guided practice:* Students should have the opportunity to attempt activities and ways of showing understanding with guidance from the teacher. This allows the student to practice what has been taught, but teacher guidance ensures that misconceptions, errors, and any other difficulties are caught quickly and addressed before the student has reinforced and practiced the errors.

*Activities to develop understanding.*

To develop an understanding of the ten to one ratio between the digits, begin by giving the student a number of base-ten blocks representing place value concepts to one hundred (flat [100], rod [10] and unit [1]). Ask the student to show how many unit blocks are needed to make the rod, then the flat. Record information so that it is available for later referral by the student. Have the student show how many rods are needed to build
flat. Once the student has seen, demonstrated and discussed the relationship between the blocks, introduce the names of the blocks (the names of each place) by tying them to what the student has discovered. The unit block is named “one”, the rod is named “ten” because it takes ten unit blocks to build it, the flat is named hundred because it is built with one hundred unit blocks.

To develop an understanding that the digits in numbers each represent a different quantity, begin by having the student and teacher each take a handful of ones and tens blocks (a bank) to work with. The teacher introduces the place value table labeled with the names of the columns (See figure 1). The teacher models the activity by building quantities and then recording the amount built in the place value chart. Then the student is given the opportunity to build and record various quantities.

In order to develop an understanding that the position of a digit determines its value use a set of small flashcards of the digits 0 to 9, a place value chart, and a set of base-ten blocks. The teacher will create a number. He or she will model how the number is said verbally and how it is recorded in the place value chart (see Figure 2). Then using the information from the chart, will build a concrete model of the number using the base-ten blocks. The digits can then be reversed and the same process repeated. Keeping both concrete models and the recorded numerals in the place value chart, the teacher can talk about how the numbers in the chart are the same and how they
are different by verbalizing how the concrete models differ. The same process can be repeated using a different arrangement of selected digits, and the student is given the opportunity to work through the same process. As well, the student should be asked to record their information both within a place value chart and as a standard number, in order to ensure that he or she understands that a number in standard form represents the same quantity as a number written using place value notation or a place value chart (Liedtke, 1993).

It often seems simple to tell children that they cannot have more than nine blocks per column, but without an understanding of why, this information becomes just one more rule to be forgotten or confused. Instead, to develop an understanding that there can only be one digit per place, a structured learning activity is beneficial. Provide both teacher and student with a “bank” of blocks and using the verbal name of a quantity, the teacher models building a quantity in more than one way using more than one type of block (see figure 3). Then ask the student to build another quantity using base-ten blocks in more than one way. Using a place value chart, compare how each quantity is recorded. Then record the number outside of the chart so that the student can see how the number would be written in its conventional form. Ask the student which form is correct, and what they think could be done to the incorrect model in order to change the way the incorrect numeral is written. Link this with their previous knowledge of how to write numbers less
than 100. Repeat this activity a number of times, each time asking the student to consider which way of writing numbers is correct, and to find a way to correct the concrete model that did not result in a conventionally written number. Students should be led to understand that numbers are written correctly when there is no possibility of trading for a larger block. (Activity adapted from Liedtke, 1993).

Once students understand that digits have different values based upon their place value position, this knowledge can be used to build understanding of expanded form. Provide the student with a “bank” of base-ten manipulatives and ask him to build a selected number. Separate the blocks into their different place value groups and write down the actual value of each pile of blocks. Addition can be modeled by bringing the groups of blocks together into a single grouping once again (See figure 4). Have the student repeat the activity with the guidance of the teacher until the student can successfully separate, record and demonstrate an understanding of implied addition.

Understanding that numbers have a relative size that is determined by the arrangement of digits can be taught using base-ten blocks. Provide the student with a “bank” of base-ten blocks. The teacher produces a list of unique numbers (numbers that do not share any common digits, i.e. 15, 43, 27, 86) and models building each one. Then the teacher
models the thought process for ordering numbers, by verbalizing the comparison of two of the numbers (see Figure 5). Each of the remaining numbers is compared and then placed in order by size. The student is then provided with the opportunity to work through the process with a variety of different unique numbers. Once students understand these concepts with unique numbers, they can begin to consider the order and relative size of non-unique numbers (i.e. 63, 36, 34, 43 etc). The student can also be introduced to the idea of a number line (0, 1, 2, 3, 4, etc) and work to build number lines with the numbers he or she is asked to compare.

Games to Reinforce and Practice Concepts and Skills involved in Place Value:

**First to 100 (or 1000).**

One six sided die or spinner is needed. Each student is provided with a “bank” supply base-ten blocks. If they are playing to or from 100, each child will need one “hundred” block, ten “ten” blocks and sixteen “one” blocks. If they are playing to or from 1000 each will need one “thousand” block, ten “hundred” blocks, twelve “ten” blocks and sixteen “one” blocks. In turn, each student rolls the dice and builds the quantity that he or she rolled by removing blocks from his or her “bank”. In each turn, the student adds to his or her quantity according to what has been rolled. As students build, they are forced to regroup in order to have blocks available. The winner is the first one to reach the preset quantity. This game can also be played as “First to Zero”.

<table>
<thead>
<tr>
<th>Hundreds</th>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rolled 5 in the turn. Rolled 4 in the first turn.</td>
<td>5</td>
<td>+ 4</td>
</tr>
<tr>
<td>second turn. Rolled a 6</td>
<td>9</td>
<td>+ 6</td>
</tr>
<tr>
<td>Rolled a 5. “one” blocks only have 16 so</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>“one” blocks must trade before she can add 5 more blocks.</td>
<td>+ 5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Students begin with a preset quantity and instead of adding blocks each time, must remove blocks from their pile. Again, regrouping is required in order to have necessary blocks available. The game can be extended by requiring students to keep a running record of their dice rolls and trading (see figure 6).

**Lucky Greatest Number**

Each child is given a game sheet. (see figure 7). The goal of the game is to make the greatest possible number. The teacher rolls a single dice and calls out the digit rolled. Each student must decide in which box to place the digit. Once the digit is placed it cannot be moved. Each digit must be placed before the next dice roll. The two right most boxes are “throw away” boxes, where students can place digits they do not wish to use. The teacher continues to roll the dice until five digit have been rolled. The student who has build the largest number wins the round. This game can be adapted in many ways including changing the size of numbers, playing for least number or a number closest to a given quantity (adapted from Liedtke, 1993).

**Interventions for Developing Number Operations:**

Students with learning difficulties in mathematics need to master the basic mathematical skills of addition, subtraction, multiplication and division (Van Kraayenoord and Elkins, 2004). However, most students understand the basic action associated with addition and subtraction. Difficulty often begins at the time when regrouping is introduced. The difficulties with regrouping are addressed by ensuring a conceptual understanding of
place value and then demonstration with, and practice using, manipulative materials such as
base-ten blocks to practice solving addition and subtraction questions. These materials are
also helpful in modeling the algorithms for partial product multiplication and long division.

Beginning multiplication and division can be taught effectively using manipulative
materials and ample varied real life story situations such as “Sue is having a party for six
people and she needs four party favors for each person. How many party favors does she
need?” or “Pete’s mom bought a bag of twenty candies for him and his four friends. If they
share all the candies, how many will each get?”. Modeling the use of manipulatives such
as counters is also beneficial. Students need to develop a mental image of addition as
separate groups of objects being put together into one large group and subtraction as either
the removal of quantity from a group, or as comparison between two quantities. A mental
image for multiplication involves a number of groups, each containing the same quantity,
and the total amount being counted. Division is when a larger quantity is split up into
equal size groupings, and the amount to be considered is either the quantity in one group or
the number of groups. Having an understanding of what each operation does, through use
of familiar context story problems, will help students to recognize the application of
operations in real life situations (Liedtke, 1991). To ensure that students develop a strong
conceptual understanding of each operation, students should be given plenty of time to
practice and the opportunity to solve many different story problems.

The need for quick and efficient recall of basic math facts is also important for
students and in particular, students with difficulties in mathematics benefit from being
taught strategies for finding answers to calculation problems. Research recommends
teaching counting on, grouping for ten, and adding doubles as specific ways to aid in quick recall of facts. However, these strategies are only appropriate for addition and subtraction. To aid with the recall of multiplication and division facts, teaching of patterns, teaching relationships between specific facts and providing opportunities for students to practice and memorize as many facts as possible are useful approaches.

Counting on can be taught for both addition and subtraction, as students can use the same strategy for both operations. This is most useful when the student still counts all objects when adding. The teacher can model using manipulatives such as counters or a number line to build quantities and then verbalize “I know there are ** in this pile so I don’t need to count it again. I can just count the rest”. The same strategy can be used when subtracting, but instead the student would count down as they removed counters.

Teaching doubles requires students to memorize the facts for 1+1, 2+2, 3+3 and so on, up to 9+9. Then students can be taught to recognize when a basic facts question closely resembles a question involving doubles. Through modeling and practice, the student comes to understand that 8+9 can be solved by remembering 8+8, and then adding one more. In much the same way, grouping for ten requires students to know basic facts to ten (all the addition questions for which the answer is 10) and to understand the pattern for adding on to ten. The student is lead to understand that in the question 8+6, the 6 can be broken down into 2 +4 because 8+2=10 and then 10+4=14. However, the strategy of grouping to ten requires a number of steps and so taxes the working memory. Hence, trying to learn this strategy may be very frustrating and difficult for some students. Neither the “adding doubles” or “grouping to ten” are particularly useful for teaching quick
recall of subtraction facts.

Teaching multiplication and division strategies requires awareness of number patterns. Most students can be taught to rote count by twos, fives and tens, and this knowledge can be linked to multiplication facts. Students can count on their fingers using skip counting patterns. For example, to find the answer to $4 \times 2$, the student can hold up four fingers and count 2, 4, 6, 8. Most students find the two, five and ten groups of facts easier to memorize than others, and often have a number of the 45 basic multiplication facts (up to $9 \times 9$) memorized. This knowledge can also be used. If students know a math fact such as $5 \times 6$, then they also know $6 \times 5$, and can use counting on (adding 6) to find the answer to $6 \times 6$ or counting down (subtracting 6) to find the answer to $6 \times 4$ (adapted from Liedtke, 1991). Division facts often seem to be the most challenging to learn and are best learned when associated with the related multiplication fact.

Since memorization continues to play a role, even when teaching strategies, it is important to ensure that ample pleasant practice is provided. One way to do this is through computer assisted instruction. There are a number of widely available commercial products which provide repetitive fact presentation embedded in a game format. Breaking the facts to be learned down into smaller groups, such as the four times tables or the facts $4 \times 1$, $4 \times 2$, $4 \times 3$, $4 \times 4$, and $4 \times 5$, focuses the student’s attention on a specific group of facts and provides greater opportunity to explore the relationship between the small group of facts. There are a number of simple card and dice games that can be adapted as needed to match the group of facts the student is learning. Bingo games, where the boxes are filled with either questions or answers (see figure 8), and the teacher or another student calls out the
matching answers or questions, is one game many children enjoy. Another game uses the same bingo format of several rows of boxes. Students can play in groups of two or more.

Each student has their own sheet and fills in the answers to the set of questions that are being learned. They then use a single dice (or modified cube) to roll questions. The answer is coloured in and the first student to colour in a row, column or diagonal wins the game. These games can be adapted to match the level, knowledge, and needs of individual students. Games, such as those described above, still require students to use other strategies to determine unknown answers, and for students who have been taught some of the above strategies, the use of varied strategies and rehearsal will help build a mental web of associated information, which in turn will result in improved math fact recall. As well, many early childhood education teachers utilize rhyme and song to teach ideas such as the names of the letters of the alphabet, letter sounds and rhyming patterns. This same technique has been capitalized on to help students memorize multiplication and division facts. There are a number of commercial internet sites which sell compact discs of catchy songs which are intended to aid students in memorizing their math facts.
CHAPTER 4
SUMMARY

As a result of recent interest in early numeracy by the school districts and the British Columbia Ministry of Education, the author desired to explore the current research into this area and then apply that research to assisting older students who struggle with mathematics. The desire was to develop interventions in a way that would positively affect the development of numeracy in students with mathematical learning difficulties, in much the same way that early literacy interventions can be utilized effectively with older elementary students in order to assist them in developing their literacy skills. As such, the issue of numeracy development was to form the core of research and the purpose of the study was to answer the following questions:

1. What are the key elements of number concepts identified as presenting the greatest challenge to students with mathematical learning difficulties?
2. What are the key components of successful mastery of number operations?
3. What general contributing factors to learning difficulty have the greatest impact for those with mathematical learning difficulties and how do these affect the acquisition of number concept and operations?
4. What instructional strategies and activities are most effective for assisting students with mathematical learning difficulties develop mathematical skills and concepts involved in the area of number concepts and operations in order to support effective numeracy development?

An issue that became apparent early on was the lack of North American research about numeracy development. While position papers such as the NCTM Standards (1989)
exist, there has been little work published in journals regarding the “how to” or “what to” portions of numeracy development. While Australia, New Zealand and the United Kingdom have begun working in this area, much of what is available focuses on improving general classroom teaching practice, improving the mathematical learning of all students. However, this has resulted in less attention being given to what happens for students who continue to experience significant difficulties in the classroom beyond the early primary years. As well, much of what is currently available is philosophical in nature, rather than practical in its application. This led to a review of the research on interventions for particular mathematical ideas, particularly those that appeared related to values espoused in commentary on numeracy.

The difficulty in finding current research into specific attributes of numeracy and effective numeracy development practices has made finding the answers to the research questions challenging. It is the author’s opinion that further study into the common skills, knowledge and concepts possessed by numerate individuals of all ages is an important area for further exploration by researchers.

**Question 1:** What are the key elements of number concepts identified as presenting the greatest challenge to students with mathematical learning difficulties?

The research explored in this paper identified counting skills and place value concepts as posing great challenges for students in acquiring proficiency and understanding within the area of number concepts. Classification, comparison and ordering appear to offer increased challenges to students with learning difficulties. Many aspects of place value also challenge students, this can be observed in the often abstract tasks involved in learning place value, and the difficulty students often demonstrate when applying place
value ideas to basic operations (regrouping).

**Question 2:** What are the key components of successful mastery of number operations?

The areas of concern regarding the teaching and learning of number operations do not center around teaching students how to perform operations or algorithms, but instead are concerned more with basic facts recall, strategy development and conceptual understanding of the actions associated with each operation. Research recognizes that one of the primary difficulties for students with mathematical learning difficulties is the application of mathematical knowledge to real life applications and so there is a strong emphasis on the development of conceptual understanding linked with procedural knowledge.

As well, basic fact recall continues to be a focus point in both instruction and research. Assisting students who demonstrate poor strategy development and memory problems to access basic information quickly, is considered by many (Geary, 2004; Miller, Butler and Lee, 1998; Tournaki, 2003; Woodward and Montague, 2002) as essential for success in mathematics.

**Question 3:** What general contributing factors to learning difficulty have the greatest impact for those with mathematical learning difficulties and how do these affect the acquisition of number concept and operations?

Research has identified a number of contributing factors to mathematical learning difficulties which range from working memory and retrieval difficulties to organizing a page in a readable and workable manner (Keeler and Swanson, 2001; Kroesbergen, van Luit and Naglieri, 2003; Miller and Mercer, 1997; Steele, 2002; van Kraayenoord and
Elkins, 2004; van Luit and Schopman, 2000; Woodward and Montague, 2002). In general, these factors are not linked directly to a failure or inability to learn, but rather are viewed as challenges to be addressed. These factors are presented as contributing to errors in assignments rather than contributing to poor conceptual development. Working memory difficulties may contribute to a student with mathematical learning difficulties having difficulty remembering steps in operations, or recalling needed basic facts, but should not affect a student’s ability to understand a mathematical idea. Visual-perceptual difficulties may contribute to errors in calculations because students are unable to read and follow the work done on a page, and so make many calculation errors, but this does not contribute to difficulties understanding the modeling done by the teacher. Attentional difficulties may make focusing on important information more difficult, but does not necessarily mean that the student cannot understand the important information if their attention is focused.

Poor teaching also forms a significant barrier to mathematics learning. Students with difficulties in the areas of working memory, attention or organization are more likely to develop mathematical understanding and numeracy skills when they have the benefit of appropriate instruction. A few students learn despite poor teaching, very few students with mathematical learning difficulties progress and develop the mathematical skill and understanding necessary for numeracy under such circumstances.

All of this has a cumulative negative effect on mathematical performance and understanding, however, the recommendations for interventions are not significantly different than what is identified as effective mathematical teaching practice for average students. Students with mathematics learning difficulties appear to benefit from the same strategies and approaches which are effective with students with average mathematical
learning. Students with learning difficulties appear to need more of the good things: structure, modeling, practice and time.

**Question 4:** What instructional strategies and activities are most effective for assisting students with mathematical learning difficulties develop mathematical skills and concepts involved in the area of number concepts and operations in order to support effective numeracy development?

Research has identified a number of effective strategies for teaching mathematical concepts to all students. These include using a concrete to abstract teaching process, using manipulative materials, modeling procedures and activities, and providing guided and independent practice (Booker, 1999; Cass, Cates, Smith and Jackson, 2003; Fuchs and Fuchs, 2001; Miller, Bultler and Lee, 1998; Miller and Mercer, 1997; Smith and Gellar, 2004; Steele, 2002; Westwood, 2003). There is an increasing awareness of the necessity for appropriate, paced instruction and ample practice time for students. However, the recommended audience for these instructional strategies and interventions is not limited to students within the parameters of special education or identified as having mathematical learning difficulties. The instructional emphasis for students with learning difficulties is the same as that for the general education student. General education students may understand ideas more quickly, develop strategies independently, or develop an understanding of patterns and relationships incidentally, despite poor instruction, but appropriate, structured and effective instruction is essential for students with mathematical learning difficulties.

**General Concluding Comments**

The primary purpose of this paper was to develop an understanding of early
numeracy development and how those ideas could be used to provide effective interventions for older elementary students with mathematical learning difficulties. Overall, the body of research explored, while providing a superficial understanding of some of the issues involved in developing numeracy skills in students with mathematical learning difficulties, has provided only a limited scope for instructional intervention. While this paper discusses the current mathematics teaching philosophy and recommended practice, it was difficult to find information that focused primarily on numeracy and how it can be developed. It appears that research has often been conducted in order to identify successful remediation activities for the parts of the curriculum that students have not mastered rather than to identify what mathematical skills are needed in order for one to become a numerate individual. In order for educators to effect instructional change and provide appropriate support for the development of numeracy in mathematically challenged students, research is needed which focuses on identifying both the important mathematical elements of numeracy and the practical interventions which are proven to help students with mathematical learning difficulties understand and master necessary mathematics and thereby develop numeracy. Research into reading and writing has become focused in recent years on how to support students in becoming good readers and writers by melding practices and philosophy from all sides, and instructional practice now reflects the change of focus from whole language versus phonics to whole language and phonics. It is time for instructional practice in mathematics to follow suit.
References


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